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Crystal families and systems in higher dimensions, and geometrical symbols of their point groups. I. Crystal families in five-dimensional space with two-, three-, four- and sixfold symmetries

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The aim of this paper and of the following one [Weigel, Phan & Veysseyre (2008). *Acta Cryst.* A**64**, 687–697] is to complete the list of the Weigel–Phan–Veysseyre (WPV) symbols of the point groups of space E^5 that was started in previous papers and in two reports of an IUCr Subcommittee on the Nomenclature of *n*-Dimensional Crystallography. In this paper, some crystal families of space E^5 are studied. The cells of these are right hyperprisms with as a basis either two squares, or two hexagons, or a square and a hexagon. If the basis is made up of two squares, the two families are the (monoclinic di squares)-al family (No. XVI) and the (di squares)-al family (No. XIX). If the basis is made up of two hexagons, the two families are the (monoclinic di hexagons)-al family (No. XVII) and the (di hexagons)-al family (No. XXI). If the basis is made up of one square and one hexagon, the family is the (square hexagon)-al family (No. XX). In order to link space E^5 to spaces E^2, E^3 and E^4, some results published in previous papers are recalled. In fact, most of the symbols of the point groups of space E^5 can be deduced from the symbols of the four, six and 23 crystal families of spaces E^2, E^3 and E^4, respectively.

1. Introduction

In the paper by Veysseyre & Veysseyre (2002), a method for obtaining the crystallographic points groups of five-dimensional space was described and the number of all the subgroups of each of the 32 crystal-family holohedries of space E^5 was found. The 955 point groups [this number was given by Plesken (1981)] were defined by all their elements. In a subsequent paper (Veysseyre *et al.*, 2002), the Weigel–Phan–Veysseyre (WPV) symbols (Weigel *et al.*, 1987) of 15 families, from family I to family XV, were given together with a detailed study of a specific family, No. XI, called the (hexagon oblique)-al family. In the present paper we assign symbols to the point groups of the following crystal families: the (monoclinic di squares)-al family (No. XVI), the (di squares)-al family (No. XIX), the (monoclinic di hexagons)-al family (No. XVII), the (di hexagons)-al family (No. XXI) and the (square hexagon)-al family (No. XX). This study complements the results given in two reports of the IUCr Subcommittee on Nomenclature of *n*-Dimensional Crystallography (Janssen *et al.*, 1999, 2002).

To assign a WPV symbol to each point group, two complementary methods are used.

(1) A computer-analysis method written by H. Veysseyre (Veysseyre *et al.*, 2002) can be used. This gives for each crystal family the number of point groups of the holohedry and for each group the symmetry elements; for example, the number of rotations of order 2, 3 and so on.

(2) A geometrical study of the family cells can be carried out. These cells are either polygons of space E^2, for example a square or hexagon, or polyhedra of space E^3, for example a cube. Hence their symmetry elements and the symbols that describe them are well known.

We recall that WPV notation gives a name to each crystal family that explains the way the cell is built and gives a symbol to each point group. Therefore, owing to this name, it is possible to build the family cell, and to give the number and the nature of the parameters that define it. For instance, di orthogonal squares means that the cell belongs to space E^4, it is built from two unequal squares which belong to two orthogonal spaces and it depends on two length parameters, *i.e.* the lengths of the sides of the squares. If we add the suffix 'al', the cell of the family is a right hyperprism. The family (di orthogonal squares)-al belongs to space E^5, its cell is a right hyperprism based on the cell of the di orthogonal squares family and it depends on three parameters, *i.e.* the sides of the two squares and the height of the hyperprism. The WPV symbols of the holohedries of these two families are easily found: $4mm\perp 4mm$ for the first and $4mm\perp 4mm\perp m$ for the second. It is then possible to describe these two groups precisely. In §2, a detailed explanation of WPV notation is given.

Finally, in Appendix *A*, we explain how the symbols of the point groups of the square triclinic, (square oblique)-al and square orthorhombic crystal families of space E^5 can be deduced from the symbols of the point groups of the square, tetragonal and square rectangle crystal families of spaces E^2, E^3 and E^4, respectively. The same study can be carried out for the hexagon, triclinic, (hexagon oblique)-al and hexagon orthorhombic crystal families.

In the following paper (Weigel *et al.*, 2008), a study of crystal families whose cells are built from one or several cubes is reported.

We recall that, by convention, the crystal-family numbers are Roman numbers (Weigel & Veysseyre, 1993; Plesken, 1981; Plesken & Hanrath, 1984) or Arabic numbers (Janssen *et al.*, 1999, 2002; Brown *et al.*, 1978) for crystal families of space E^4, and Roman and Arabic numbers (Janssen *et al.*, 1999) for crystal families of space E^5.

2. WPV point-group symbols of space E^n recalled

2.1. Main rules for assigning a WPV symbol to a point group

These main rules are explained in Veysseyre *et al.* (1998, 2002).

All the groups studied in this paper describe the symmetries of crystal structures, so mathematical symbols are not always used. For example, there is only one abstract group of order 2. The isomorphic groups of order 2 that describe the symmetries of crystal structures have as a symbol 2 if they describe a rotation of order 2, *m* if they describe a reflection, and $\overline{1}$, $\overline{1}_4$ and $\overline{1}_5$ if they describe *homotheties* in spaces E^3, E^4 and E^5, respectively.

Where possible, Hermann–Mauguin notation is used (*International Tables for Crystallography*, 1995) *i.e.* 4*mm* for a square, $m\overline{3}m$ for a cube and so on.

The WPV symbol gives without ambiguity the group order and all these elements, and permits recognition of the family. When the name of the family is known, hence the cell, it is easy to find the geometrical supports of the generators that appear in the symbol and those of all the elements of the group. The geometrical supports are defined in Weigel *et al.* (1987). However, it can be useful to give the geometrical supports of the generators for a few groups, so we have given them in the corresponding tables.

If the point group has at least one reflection, the letter *m* must appear in the symbol and we adopt the Hermann–Mauguin rule giving the reflection number of the group. For example, group 3*m* has three reflections *m*, group 6*mm* has 2 × 3 = 6 reflections *m* and group 4*mm* has 2 × 2 = 4 reflections *m*. Group $\overline{4}2m$ has three rotations 2 and two reflections *m*.

If the group is the product of two or more subgroups, different symbols are used according to the nature of the product. The symbol ⊥ must be used when the two subgroups belong to two orthogonal subspaces; then the product is a direct and orthogonal product of groups. The symbol × is used for a direct product of groups without any geometrical relation. Finally, the symbol . is used for a non-direct product. Thus the symbol ⊥ enables us to define without ambiguity the generators and their supports, and therefore all the group elements.

2.2. WPV symbols of cyclic groups

These are generated by different crystallographic operations in space E^n.

(i) Operations of order 2: *m* ($n \geq 1$), 2 ($n \geq 2$), $\overline{1}$ ($n \geq 3$), $\overline{1}_n$ ($n \geq 4$).

(ii) Operations of order 4: 4 ($n \geq 2$), $\overline{4}$ ($n \geq 3$), 42 and 44 ($n \geq 4$), $\overline{\overline{4}} = 4\times\overline{1}_5$ and $\overline{\overline{44}} = 44\times\overline{1}_5$ ($n \geq 5$).

(iii) Operations of order 3 and 6: 3 and 6 ($n \geq 2$), $\overline{3}$ ($n \geq 3$), 32, 33, 34, 36, 62, 64, 66 ($n \geq 4$).

(iv) Operations of order 8: [8], which is the group generated by the double rotation $8^1 8^3$ ($n \geq 4$). This group belongs, for example, to the monoclinic di iso squares family (No. 14_1 of space E^4).

(v) Operations of order 5 and 10: Group [5] is generated by the double rotation $5^1 5^2$ and group [10] is generated by $10^1 10^3$ ($n \geq 4$). Group [10] belongs, for example, to the decadic family (No. 15_1 of space E^4).

(vi) Operations of order 12: 46, [12] ($n \geq 4$) and $\overline{\overline{43}} = 43\times\overline{1}_5$ ($n \geq 5$). Group [12], which is generated by the double rotation $12^1 12^5$ belongs, for example, to the monoclinic di iso squares family (No. 16_1 of space E^4).

2.3. Different ways of describing the same group

In Janssen *et al.* (1999), some groups have two symbols. In fact, there are two different ways of describing the same group; these symbols are denoted as either symbols or alternative symbols in Janssen *et al.* (1999, Table 3, p. 768). For example:

(i) System 08_1: $4m(12) = 4.\overline{1}$ because 4.1 = 4 and $m.2 = \overline{1}$; in the same way $4m(22) = 42.\overline{1}$. In fact, this group could have the symbol $42\overline{1}m$, which is a Hermann–Mauguin notation similar to $\overline{4}2m$, but we prefer the shortest symbol 42.*m* which gives the order of the group (8 = 4 × 2) and all its elements.

(ii) Systems 12_1 and 12_2: $4m(m2) = \overline{4}.\overline{1}$ and $4m(2m)$ = 42.2. The symbol 42 is explained in §3.4.1.

(iii) System 05_1: the holohedry symbol is 44 instead of 4(4).

(iv) System 15_1: the holohedry symbol is [10].2 instead of $10m(10^3m)$; indeed, this group is generated by the operation $(10^1 10^3).2_{xy}$ because $m_x m_y = 2_{xy}$.

(v) System 17_1: the WPV symbol of the fifth point group, $\overline{4}.\overline{4}$, is simpler than (4*m*1)*m*42, but in fact these two symbols are similar because $4_{xy}.m_z = \overline{4}_{xyz}$ and $m_x.1.42_{ztxy} = \overline{4}_{zty}$. In the same way, 4*m*1(412) = 44*m*2 (in Hermann–Mauguin notation), but we prefer 44.*m*.

(vi) System 17_2: similar remarks can be made for the point groups of this system.

In conclusion, if two symbols can be used to describe the same group, we adopted the simplest symbol because it also gives the order of the group and its elements.

Table 1
From the diclinic di squares family (system 05_1, space E^4) to the (diclinic di squares)-al family (No. XII, space E^5).

The double rotation 44 belongs to the space ($xyzt$) and the reflection m to the axis u. $\overline{\overline{44}}$ is the product of the double rotation 44 and *homothetie* $\overline{1}_5$ and generates a cyclic group of order 8. The order of the group $44\perp m$ is $8 = 4 \times 2$.

System 05_1 (1 point group)	Family XII (3 point groups)		
E^4	E^4	E^5	$E^4 + E^1$
44 (hol.)	44	$\overline{\overline{44}}$	$44\perp m$ (hol.)

Table 2
From the monoclinic di squares family (system 10_1, space E^4) to the (monoclinic di squares)-al family (No. XVI, space E^5).

The double rotation 44 belongs to the space ($xyzt$) and the reflection m to the axis u. Group 2 acts in the space (xz) and *homothetie* $\overline{1}$ in the space (xzu). The order of the group $(44.2)\perp m$ is $16 = 4 \times 2 \times 2$.

System 10_1 (1 point group)	Family XVI (4 point groups)			
E^4	E^4	E^5	E^5	$E^4 + E^1$
44.2 (hol.)	44.2	$\overline{\overline{44}}.2$	$44.\overline{1}$	$(44.2)\perp m$ (hol.)

3. The di squares families of spaces E^4 and E^5

The aim of this section is to explain how to obtain WPV symbols of the 59 point groups of the (di squares)-al family (No. XIX) of space E^5 from the point groups of the following three families: the di squares family of space E^4, and the (monoclinic di squares)-al and the (diclinic di squares)-al families of space E^5. The holohedry of the (di squares)-al family has as a WPV symbol $4mm\perp 4mm\perp m$ and its order is $128 = 8 \times 8 \times 2$.

Holohedry is abbreviated as (hol.) in all the tables.

3.1. The WPV point-group symbols of the three di squares families of space E^4

In space E^4 there are three di squares families. These squares belong to spaces (xy) and (zt).

(i) The diclinic di squares family (system 05_1). Its holohedry is group 44 of order 4, the only point group of this family.

(ii) The monoclinic di squares family (system 10_1). Its holohedry is group 44.2 of order 8, the only point group of this family.

The point groups of systems 05_1 and 10_1 are listed in the first columns of Tables 1 and 2, respectively.

(iii) The di squares family. This family splits into two systems:

(a) System 17_1. This has five point groups and group 44.mmm of order $32 = 4 \times 2 \times 2 \times 2$ for holohedry. This group has four reflections m.

(b) System 17_2. This has six point groups and group $4mm\perp 4mm$ of order $64 = 8 \times 8$ for holohedry.

The point groups of systems 17_1 and 17_2 are listed in the first column of Table 3. We denote by G_4 the set of 11 point groups of the di squares family of space E^4. This set is divided into two subsets G_4' and G_4'':

(i) G_4' is the subset of the five point groups of system 17_1 *i.e.* groups 44×2, 44.mm, 44.222, 44.mmm and $\overline{4}.\overline{4}$. G_{44}' is the subset of the four point groups of the set G_4' that have the group 44 as the first factor.

(ii) G_4'' is the subset of the six point groups of the system 17_2, *i.e.* groups $4\perp 4$, $4.\overline{4}$, $4.\overline{4}2m$, 4.422, $4mm\perp 4$ and $4mm\perp 4mm$.

3.2. The (diclinic di squares)-al family (No. XII) of space E^5

The cell of the (diclinic di squares)-al family is a right hyperprism whose basis is the cell of the diclinic di squares family of space E^4. We add an axis u orthogonal to space E^4. Consequently, we obtain on the one hand reflection m_u and thus group $44\perp m$ (the holohedry of this family is of order $8 = 4 \times 2$) and, on the other hand, *homothetie* $\overline{1}_5$ (the product of *homothetie* $\overline{1}_{xyzt}$ belonging to the holohedry of the diclinic di squares family and reflection m_u) and thus group $\overline{\overline{44}}$. With group 44, we obtain the three groups of family XII (these groups are listed in columns 2, 3 and 4 of Table 1).

3.3. The (monoclinic di squares)-al family (No. XVI) of space E^5

To study this family, a similar method gives the four point groups listed in columns 2, 3, 4 and 5 of Table 2. These groups are obtained from group 44.2 of the monoclinic di squares family of space E^4. In fact, as before, we introduce reflection m_u which gives group $(44.2)\perp m$ (the holohedry of family XVI is of order $16 = 4 \times 2 \times 2$) and *homothetie* $\overline{1}_5$, which gives groups $\overline{\overline{44}}$ and $44.\overline{1}$; the *homothetie* denoted $\overline{1}_{xzu}$ is the product of the rotation 2_{xz} and the reflection m_u.

3.4. The (di squares)-al family (No. XIX) of space E^5

Family XIX splits into two subfamilies:

(i) Subfamily XIXa. This has 25 point groups and group $(44.mmm)\perp m$ of order $64 = 4 \times 8 \times 2$ for holohedry.

(ii) Subfamily XIX. This has 34 point groups and group $4mm\perp 4mm\perp m$ of order $128 = 8 \times 8 \times 2$ for holohedry. The definition of the subfamily is given in Veysseyre *et al.* (2002).

The cell of family XIX can be considered:

(i) as a right hyperprism whose basis is the cell of the di squares family (space E^4). After adding an axis u orthogonal to space E^4, we construct two point operations, a reflection m_u and a *homothetie* $\overline{1}_5$ as previously;

(ii) or as the product of a square cell and a tetragonal cell belonging to two orthogonal spaces.

3.4.1. From system 17_1 of space E^4 to subfamily XIXa of space E^5. The 25 point groups of subfamily XIXa are listed in Table 3 (first part, columns 2 to 6). We find:

(i) the five groups g_4' of the subset G_4' (second column);

(ii) the five groups $g_4'\perp m$ (third column);

Table 3
From the di squares family (No. 17, space E^4) to the (di squares)-al family (No. XIX, space E^5).

The double rotation 44 belongs to the space ($xyzt$) and the reflection m to the axis u. The double rotations 42 of the group 42.42 belong in the spaces ($xyzu$) and ($ztxu$), respectively. Group 2 acts in the space (xy) after the symbol '×' and in the space (xz) after the symbol '.'; *homothetie* $\bar{1}$ acts in the space (xyu) after the symbol '×' and in the space (xzu) after the symbol '.'; *homothetie* $\bar{1}_4$ acts in the space ($xyzt$) after the symbol '.'. The order of the group $\bar{\bar{4}}.\bar{4}2m$ is 32 = 4 × 8. The order of the group $422.\bar{4}2m$ is 64 = 8 × 8.

System 17_1 (5 point groups)	Subfamily XIXa (25 point groups)				
E^4	E^4	$E^4 + E^1$	E^5	E^5	E^5
44×2	44×2	$(44\times2)\perp m$	$(44\times2).\bar{1}_4$ $(44\times2).\bar{1}$	$\overline{\overline{44}}\times2$ $(\overline{\overline{44}}\times2).\bar{1}_4$	$44\times\bar{1}$ $(44\times\bar{1}).2$
$44.mm$	$44.mm$	$(44.mm)\perp m$	$(44.mm).\bar{1}_4$	$\overline{\overline{44}}.mm$	
44.222	44.222	$(44.222)\perp m$	$(44.222).\bar{1}_4$	$\overline{\overline{44}}.222$	
$44.mmm$ (hol.)	$44.mmm$	$(44.mmm)\perp m$ (hol.)		$\overline{\overline{44}}.mmm$	
$\bar{4}.\bar{4}$	$\bar{4}.\bar{4}$	$(\bar{4}.\bar{4})\perp m$	$(\bar{4}.\bar{4}).\bar{1}_4$		$\bar{4}.42$ $\bar{4}2m.42$ 42.42

System 17_2 (6 point groups)	Subfamily XIX (34 point groups)				
E^4	E^4	$E^4 + E^1$	$E^2 + E^3$	E^5	E^5
$4\perp4$	$4\perp4$	$4\perp4\perp m$			$(4\perp4).\bar{1}$
$4.\bar{4}$	$4.\bar{4}$	$(4.\bar{4})\perp m$		4.42 $4.42.m$	$4.\bar{4}.\bar{1}$
			$4\perp\bar{4}$		$(4\perp\bar{4}).\bar{1}$
				422.42 $\bar{4}.\bar{\bar{4}}$ $\bar{\bar{4}}.\bar{\bar{4}}$	$\bar{4}.\bar{\bar{4}}.\bar{1}$ $\bar{\bar{4}}.\bar{\bar{4}}.\bar{1}$
$4.\bar{4}2m$	$4.\bar{4}2m$	$(4.\bar{4}2m)\perp m$	$4\perp\bar{4}2m$	$\bar{4}.\bar{4}2m$ $\bar{\bar{4}}.\bar{4}2m$ $\bar{4}2m.\bar{4}2m$	
4.422	4.422	$(4.422)\perp m$	$4\perp422$ $4mm\perp422$	$\bar{4}.422$ 422.422 $422.\bar{4}2m$	
$4\perp4mm$	$4\perp4mm$	$4\perp4mm\perp m$	$4mm\perp\bar{4}$ $4mm\perp\bar{4}2m$		
$4mm\perp4mm$ (hol.)	$4mm\perp4mm$	$4mm\perp4mm\perp m$ (hol.)			

(iii) the four groups $g'_4.\bar{1}_4$. The operation $\bar{1}_4$ belongs to space ($xyzu$). Among the five groups of this type that are expected, group $44.mmm.\bar{1}_4$ has, in fact, another symbol: $(44.mmm)\perp m$; this group is the family XIXa holohedry of order 64 = 4 × 8 × 2 (fourth column);

(iv) the four groups isomorphic to the four groups g'_{44} of the subset G'_{44} obtained by changing operation 44 into operation $\overline{\overline{44}}$ and the group $(\overline{\overline{44}}\times2).\bar{1}_4$ (fifth column);

(v) the three groups that are the products of the group 42 (belonging to space $xyzu$) with groups $\bar{4}$, $\bar{4}2m$ and 42 (sixth column). Group 42 is the cyclic group, of order 4, generated by the double rotation $4^{+1}_{xy}2_{zt}$;

(vi) the two groups $44\times\bar{1}$ and $(44\times\bar{1}).2$ (sixth column). Group 2 is the group generated by the rotation by an angle π in the plane (zt);

(vii) the group $(44\times2).\bar{1}_4$ (fourth column).

In summary, all the WPV point-group symbols of the di squares families, *i.e.* families XII and XVI and subfamily XIXa, have been constructed:

(i) either from only one symbol denoted g_{44} of type 44 or $\overline{\overline{44}}$, *i.e.* the symbol of a double rotation of order 4 and possibly other factor groups having symmetries of order 2 only,

(ii) or from two factor point groups, each of these two groups having one and only one quaternary symmetry, *i.e.* groups $\bar{4}$, $\bar{4}2m$ of the tetragonal family, or 42 of the square oblique family. It is possible that the WPV symbol can also contain a third factor of order 2, such as m, 2, $\bar{1}$ or $\bar{1}_4$.

Conversely, every point group of space E^5 belonging to one of these types is a point group of one of the families XII, XVI or XIXa. More exactly, if the symbol is the product of two groups, different or not, of the set G_4 and possibly with a third group of the set m, 2, $\bar{1}$ or $\bar{1}_4$, this group belongs to family XIXa.

3.4.2. From system 17_2 of space E^4 to subfamily XIX of space E^5. The 34 point groups of subfamily XIX are listed Table 3 (second part, columns 2 to 6).

Table 4
From the diclinic di hexagons family (system 06_1, space E^4) to the (diclinic di hexagons)-al family (No. XIII, space E^5).

The double rotations 33 and 66 belong to the space ($xyzt$) and the reflection m to the axis u. They generate cyclic groups of order 3 and 6, respectively. The order of the group $66{\perp}m$ is 12 = 6 × 2.

System 06_1 (2 point groups)	Subfamily XIIIa (2 point groups)		Subfamily XIII (3 point groups)	
E^4	E^4	E^5	E^4	$E^4 + E^1$
33	33	$33{\times}\bar{1}_5$ (hol.)		$33{\perp}m$
66 (hol.)			66	$66{\perp}m$ (hol.)

Table 5
From the monoclinic di hexagons family (system 11_1, space E^4) to the (monoclinic di hexagons)-al family (No. XVII, space E^5).

The double rotations 33 and 66 belong to the space ($xyzt$) and the reflection m to the axis u. They generate cyclic groups of order 3 and 6, respectively. Group 2 acts in the space (xz) and *homothetie* $\bar{1}$ acts in the space (xzu). The order of the group $(66.2){\perp}m$ is 24 = 6 × 2 × 2.

System 11_1 (2 point groups)	Subfamily XVIIa (3 point groups)		Subfamily XVII (4 point groups)		
E^4	E^4	E^5	E^4	$E^4 + E^1$	E^5
33.2	33.2	$(33.2){\times}\bar{1}_5$ (hol.)		$(33.2){\perp}m$	
		$33.\bar{1}$			$66.\bar{1}$
66.2 (hol.)			66.2	$(66.2){\perp}m$ (hol.)	

All the di squares family point groups are constructed from the product of two, and only two, groups with quaternary symmetries among the following eight groups (except the four products belonging to the previous families):

(i) Four cyclic point groups (mathematical type C_4), therefore of order 4, generated by the point-symmetry operations 4, $\bar{4}$, 42 and $\bar{\bar{4}}$ belonging to spaces E^2, E^3, E^4 and E^5.

(ii) Four point groups (mathematical type D_4) therefore of order 8: $4mm$, 422, $\bar{4}2m$, $42.m$. Group 42 acts in space ($xyzt$) and reflection m in the space defined by the axis t or u.

For some groups, we add a third factor group of symmetry 2.

Conversely, every point group of space E^5 constructed this way (except for four groups) is one point group of the subfamily XIX.

Now we are in a position to list the 34 groups of the subfamily XIX. We find:

(i) The six groups g''_4 (g''_4 is an element of the subset G''_4) (second column; these belong to spaces $E^2 + E^2$ or E^4).

(ii) The six groups $g''_4{\perp}m$ (third column; these belong to spaces $E^2 + E^2 + E^1$ or $E^4 + E^1$).

(iii) The six groups $4{\perp}\bar{4}$, $4{\perp}\bar{4}2m$, $4{\perp}422$, $4mm{\perp}\bar{4}$, $4mm{\perp}\bar{4}2m$, $4mm{\perp}422$ (fourth column; these belong to space $E^2 + E^3$). Obviously, rotation 4 belongs to the space (xy) and group $\bar{4}$ to the space (ztu).

(iv) All the products of two groups taken from the previous eight groups that are not listed either in the second or fourth columns or in subfamily XIXa. There are 11 groups that can be obtained this way (fifth column).

(v) Lastly, the products of some groups by *homothetie* $\bar{1}$ in the space (xzu), which gives only five groups (sixth column).

4. The di hexagons families of spaces E^4 and E^5

4.1. WPV point-group symbols of the three di hexagons families of space E^4

In space E^4, there exist three di hexagons families, for which the hexagons belong to the planes (xy) and (zt):

(i) The diclinic di hexagons family IX, system 06_1. This has two point groups and group 66 of order 6 for holohedry. These groups are listed in Table 4 (first column).

(ii) The monoclinic di hexagons family XII, system 11_1. This has two point groups and group 66.2 of order 12 = 6 × 2 for holohedry. These groups are listed in Table 5 (first column).

(iii) The di hexagons family XVI. This family splits into three systems:

(a) System 20_1. This has four point groups and group 63.222 of order 24 = 6 × 4 for holohedry. These groups are listed in the first part of Table 6.

(b) System 20_2. This has 11 point groups and group $3m.\bar{3}m$ of order 72 = 6 × 12 for holohedry. These groups are listed in the second part of Table 6.

(c) System 20_3. This has 11 point groups and group $6mm{\perp}6mm$ of order 144 = 12 × 12 for holohedry. These groups are listed in the third part of Table 6.

In the following we explain the WPV symbols of the 4 + 11 + 11 = 26 point groups of systems 20_1, 20_2 and 20_3.

4.1.1. Subfamily 20_1. The four point groups of the system 20_1 are: the cyclic group 63 (of order 6), generated by the double rotation 63 by angles $2\pi/6$ and $2\pi/3$, for example, and the three groups resulting from the products of group 63 with groups 2, 222 and $\bar{1}_4$.

4.1.2. Subfamily 20_2. All the 11 point groups of the system 20_2 are built from the product of two factor groups from the following: 3, $3m$ (in space E^2), $\bar{3}$ (generated by the point operation $6m$), (3 2), $\bar{3}m$ (in space E^3) and 62 (in space E^4). Thus, three point groups belong to space $E^2 + E^2$ *i.e.* $3{\perp}3$, $3{\perp}3m$, $3m{\perp}3m$, and eight groups result from the product of two of them (only eight groups are formed because two different products can define the same group).

We remark that the five groups of spaces E^2 or E^3 belong to the trigonal (rhombohedral) system of the hexagonal family in the physical space E^3.

4.1.3. Subfamily 20_3. The 11 point groups of system 20_3 are built from the products of two groups. The first one is chosen from the groups 6, $6mm$ and 622, and the second one from the groups 3, $3m$, $\bar{3}$, $\bar{3}m$, 6 and $6mm$. Note that groups 6, $6mm$, 3 and $3m$ act in space E^2 whereas groups $\bar{3}$ and $\bar{3}m$ act in space E^3.

We note that 4 + 3 = 7 point groups belong to space $E^2 + E^2$, for example $6{\perp}6$. The other groups of this type belong to system 20_2. The four remaining groups are the product of a group of space E^2 with a group of space E^3 and they belong to space E^4, for example group $6.\bar{3}m$. There are only four groups of this type because two different products can define the same group, as previously.

Table 6
The di hexagons family (systems 20_1, 20_2 and 20_3, space E^4).

All these groups act in the space ($xyzt$). Rotations 3 and 6 act in the plane (xy) if they appear in the first part of the symbol and in the plane (zt) if they appear in the second part as $3m$, $6mm$. The groups $\bar{3}m$, (3 2) and 622 act in the space (xyz) if they appear in the first part of the symbol and in the space (ztu) if they appear in the second part. The double rotations 62 and 63 belong to the space ($xyzt$). They generate cyclic groups of order 6. Rotation 2 of the group 63.2 acts in the plane (xy). We have chosen the WPV symbol $3m.\bar{3}m$ for the holohedry of system 20_2 instead of $(3m\perp3m)\times\bar{1}_4$ because the first symbol is shorter than the second and is easier to generalize to space E^5.

System 20_1 (4 point groups)			
63	$63\times\bar{1}_4$	63.2	63.222 (hol.)

System 20_2 (11 point groups)										
$3\perp3$	62.3	$3m\perp3$	$3.\bar{3}m$	(3 2).3	$3m.\bar{3}$	62.(3 2)	$3m\perp3m$	$3.\bar{3}$	$\bar{3}.\bar{3}$	$3m.\bar{3}m$ (hol.)

System 20_3 (11 point groups)										
$6\perp3$	$6\perp6$	$6.\bar{3}$	$6mm\perp3$	622.3	$6\perp3m$	$6\perp6mm$	622.6	$6.\bar{3}m$	$6mm\perp3m$	$6mm\perp6mm$ (hol.)

4.2. The (diclinic di hexagons)-al family (No. XIII) of space E^5

The (diclinic di hexagons)-al family cell is a right hyperprism whose basis is the system 06_1 cell of space E^4. This family splits into two subfamilies:

(i) Subfamily XIIIa. This has two point groups and group $33\times\bar{1}_5$ of order 6 = 3 × 2 for holohedry.

(ii) Subfamily XIII. This has three point groups and group $66\perp m$ of order 12 = 6 × 2 for holohedry.

As previously, we consider an axis u orthogonal to space E^4. Therefore we obtain reflection m_u, which gives two groups, $33\perp m$ and $66\perp m$, and *homothetie* $\bar{1}_5$ (the product of *homothetie* $\bar{1}_{xyzt}$ and reflection m_u), hence the group $33\times\bar{1}_5$. With groups 33 and 66, the five point groups of the two subfamilies XIII are found. They are listed in Table 4 (second, third, fourth and fifth columns).

4.3. The (monoclinic di hexagons)-al family (No. XVII) of space E^5

The (monoclinic di hexagons)-al family is studied in the same way. It splits into two subfamilies:

(i) Subfamily XVIIa. This has three point groups and group $(33.2)\times\bar{1}_5$ of order 12 = 6 × 2 for holohedry.

(ii) Subfamily XVII. This has four point groups and group $(66.2)\perp m$ of order 24 = 12 × 2 for holohedry.

The WPV point-group symbols of families XVIIa and XVII are listed in Table 5 (second, third, fourth, fifth and sixth columns). They are obtained as the product of the two point groups of system 11_1 with reflection m_u and *homotheties* $\bar{1}$ and $\bar{1}_5$.

4.4. The (di hexagons)-al family (No. XXI) of space E^5

The (di hexagons)-al family splits into eight subfamilies denoted XXIh, XXIg, . . . , XXIa and one primitive subfamily denoted XXI. It has 116 point groups, listed in Table 7. To obtain the WPV symbols, we used the results given by the computer program, but we respected the geometrical construction of the cell and the symbols of the three di hexagons families of space E^4 were kept. We obtained:

(i) The set of the 26 groups, denoted H, of the three di hexagons families of space E^4, *i.e.* systems 20_1, 20_2 and 20_3 (first column of Table 7 for all the subfamilies).

(ii) The set of the 26 groups $h\perp m$ where h is one group of the set H (second and third columns of Table 7 for subfamily XXIb and second column of Table 7 for subfamily XXI).

(iii) Sixteen groups, the direct and orthogonal product of a group belonging to the hexagonal family (space E^3), *i.e.* groups (3 2), $\bar{3}$, $\bar{3}m$, 622, and a group belonging to the hexagon family (space E^2), *i.e.* groups 3, $3m$, 6, $6mm$. These groups are listed in the third column of Table 7: two groups for subfamily XXId, ten for subfamily XXIa and four for subfamily XXI.

(iv) A set of 14 groups, the products of two groups of the set $\bar{3}$, $\bar{3}m$, (3 2), 622, ($3\perp2$) and 62. We would have 6 + 5 + 4 + 3 + 2 + 1 = 21 groups but some of these products either define the same group or they have been listed in the first column of the table. These 14 groups are listed in Table 7 as follows: three groups for subfamily XXId (third column), one group for subfamily XXIc (second column), one group for subfamily XXIb (fourth column), two groups for subfamily XXIa (third column), and seven groups for subfamily XXI (fourth column).

(v) A set of 29 groups, belonging to space E^5, which are the product of some previous groups by a group of order 2. These 29 groups are listed in Table 7 in the columns headed E^5 for all the subfamilies. Then the set of five groups of subfamily XXIh that do not belong to the set H, *i.e.* groups $63.\bar{1}$, $63\times\bar{1}_5$, $\overline{\overline{36}}$, $\overline{\overline{36}}.2$ and $(63.2)\times\bar{1}_5$; these groups are either a cyclic group ($\overline{\overline{36}}$) or the product of a cyclic group (63 or $\overline{\overline{36}}$) with a group of order 2 or 4.

4.5. General results for families XII, XVII and XXI of space E^5

These families are numbers 10, 15 and 25 in Table 7 of Janssen *et al.* (1999); the Roman numbers are those used by Weigel & Veysseyre (1993).

Table 7
The (di hexagons)-al family (No. XXI, space E^5).

The holohedry symbol of subfamily XI is $6mm\perp6mm\perp m$; the first group $6mm$ acts in the plane (xy), the second group $6mm$ acts in the plane (zt) and the group m acts in the space u. For the groups belonging to space E^4 the generators are described in Table 6. Here we give some explanations for other groups. Rotation 2: plane (xy) for group 63.2 because it acts in space E^4 and plane (zu) for group 62.3.2, for example. *Homothetie* $\bar{1}$: space (xzu). *Homothetie* $\bar{1}_4$: space $(xyzt)$ after the symbol '×' (it is a direct product in space E^4) and space $(xyzu)$ after the symbol '.'.

Subfamily XXIh (7 point groups)			Subfamily XXIg (2 point groups)	
E^4	E^5	E^5	E^4	E^5
63	$63.\bar{1}$	$63\times\bar{1}_5$	$3\perp3$	$(3\perp3)\times\bar{1}_5$ (hol.)
	$\overline{\overline{36}}$	$\overline{\overline{36}}.2$		
63.2	$(63.2)\times\bar{1}_5$ (hol.)			

Subfamily XXIf (3 point groups)				Subfamily XXIe (3 point groups)			
E^4	$E^4 + E^1$	$E^2 + E^3$	E^5	E^4	$E^4 + E^1$	$E^2 + E^3$	E^5
			$(3\perp3).\bar{1}$				$(3\perp3).\bar{1}_4$
(3 2).3			$\{(3\,2).3\}\times\bar{1}_5$ (hol.)	$3m\perp3$			$(3m\perp3)\times\bar{1}_5$ (hol.)

Subfamily XXId (8 point groups)			Subfamily XXIc (4 point) groups	
E^4	$E^2 + E^3$	E^5	E^4	E^5
	$3\perp(3\,2)$	$\bar{3}.(3\perp2)$		62.62
	$3m\perp(3\,2)$	$62.(3\perp2)$		$(3m\perp3).\bar{1}$
$3.\bar{3}$		$3.\bar{3}.\bar{1}_4$	$3m\perp3m$	$(3m\perp3m)\times\bar{1}_5$ (hol.)
$3.\bar{3}m$		$\bar{3}.\bar{3}m$ (hol.)		

Subfamily XXIb (23 point groups)					
E^4	$E^2 + E^2 + E^1$	$E^4 + E^1$	$E^3.E^3$ (E^5)	E^5	E^5
62.3		$(62.3)\perp m$		62.3.2	$62.3.\bar{1}$
62.(3 2)		$\{62.(3\,2)\}\perp m$		$62.(3\,2).\bar{1}_4$	
		$\{3.(3\,2)\}\perp m$	(3 2).(3 2)		
	$3\perp3\perp m$	$(3.\bar{3})\perp m$		$3.\bar{3}.\bar{1}$	
	$3\perp3m\perp m$	$(3.\bar{3}m)\perp m$			
$3m.\bar{3}$		$(3m.\bar{3})\perp m$		$3m.\bar{3}.\bar{1}_4$	
$\bar{3}.\bar{3}$		$(\bar{3}.\bar{3})\perp m$		$(\bar{3}.\bar{3})\times\bar{1}_5$	
$3m.\bar{3}m$	$3m\perp3m\perp m$	$(3m.\bar{3}m)\perp m$ (hol.)			

Subfamily XXIa (23 point groups)				
E^4	$E^2 + E^3$	$E^4.E^3$ (E^5)	E^5	E^5
$6\perp3$	$6\perp\bar{3}$		$(6\perp3).\bar{1}$	$(6\perp3).\bar{1}_4$
	$3\perp\bar{3}$		$(3\perp\bar{3}).\bar{1}$	$(3\perp\bar{3}).\bar{1}_4$
622.3	$6\perp(3\,2)$	62.622		$(622.3)\times\bar{1}_5$
	$3\perp\bar{3}m$	$62.\bar{3}m$		
$6\perp3m$	$6\perp\bar{3}m$		$(6\perp3m).\bar{1}$	
	$3m\perp\bar{3}$			
	$3m\perp\bar{3}m$			
$6mm\perp3$	$6mm\perp\bar{3}$			
	$6mm\perp(3\,2)$			
$6mm\perp3m$	$6mm\perp\bar{3}m$ (hol.)			

All the point groups of the three families XII, XVII and XXI are built:

(i) either from only one group g_{33}, g_{66} or g_{63} which have two and only two symmetries of order 3 or 6, for example 63 or $\overline{\overline{36}}$, and possibly with other factors with symmetries of order 2 exclusively, for example m, 2, $\bar{1}$, $\bar{1}_4$ or $\bar{1}_5$;

(ii) or from two point groups g_6 and g_3, g_6 and g_6, or g_3 and g_3, every one having one and only one symmetry of order 3 or

Table 7 (continued)

Subfamily XXI (43 point groups)

E^4	$E^4 + E^1$	$E^2 + E^3$	$E^2.E^3$ and $E^4.E^3$	E^5
	$63\perp m$			
$63\times\bar{1}_4$	$(63\times\bar{1}_4)\perp m$			$(63\times\bar{1}_4).\bar{1}$
	$(63.2)\perp m$			$\bar{\bar{36}}.\bar{1}_4$
63.222	$(63.222)\perp m$			$\bar{\bar{36}}.2.\bar{1}_4$
$6\perp 6$	$6\perp 6\perp m$			$(6\perp 6).\bar{1}$
	$6\perp 3\perp m$			$\bar{3}.3.\bar{1}$
	$6\perp 3m\perp m$			
$6.\bar{3}$	$(6.\bar{3})\perp m$		$(3\,2).\bar{3}$	$6.\bar{3}.\bar{1}$
			$62.\bar{3}$	$62.\bar{3}.\bar{1}_4$
6.622	$(6.622)\perp m$	$6\perp 622$	$622.(3\,2)$	
	$(3.622)\perp m$	$3\perp 622$		$(3\perp 622)\times\bar{1}_5$
		$3m\perp 622$	622.622	
$6.\bar{3}m$	$(6.\bar{3}m)\perp m$		$622.\bar{3}m$	$(3\perp\bar{3}m).\bar{1}_4$
	$6mm\perp 3\perp m$		$(3\,2).\bar{3}m$	
	$6mm\perp 3m\perp m$		$\bar{3}m.\bar{3}m$	
$6mm\perp 6$	$6mm\perp 6\perp m$	$6mm\perp 622$		
$6mm\perp 6mm$	$6mm\perp 6mm\perp m$ (hol.)			

6. These factor groups belong to the sets 3, $3m$, 6, $6mm$, (3 2), 622, $\bar{3}$, $\bar{3}m$ or 62. The product of two of these groups can be a direct product or not. It is possible for other factors with symmetries of order exclusively two to appear, such as m, 2, $\bar{1}$, $\bar{1}_4$ or $\bar{1}_5$.

Conversely, every point group of space E^5 belonging to one of these two types is a point group of families XIII, XVII and XXI of this space. More exactly, if the symbol is defined from point groups g_{33} or g_{66}, it is a group of family XIII or XVII; if it is defined from only point group g_{63}, it is a group of subfamily XXIh. Finally, if this group is of the second type, it belongs to the other subfamilies of family XXI.

Remark. We have noticed an error in Table 7 of Janssen *et al.* (1999). The holohedry symbol of family 25 is $6mm\perp 6mm\perp m$, not $6mm\perp 4mm\perp m$, which is the holohedry of family 23. The metric tensor is also wrong; the correct metric tensor of family 25 is

$$\begin{matrix} a & -a/2 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & c & -c/2 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & e \end{matrix}.$$

5. The hexagon square and (hexagon square)-al families of spaces E^4 and E^5

The study of the hexagon square family of space E^4 simplifies the study of the (hexagon square)-al family of space E^5.

5.1. The hexagon square family of space E^4

This corresponds to system 18_1 in Table 3 of Janssen *et al.* (1999). The cell of the hexagon square family is built from one hexagon in the plane (xy) and one square in the plane (zt) orthogonal to the plane (xy), axes z and t also being orthogonal.

We can foresee the 22 point groups of this family, listed in Table 8 as follows:

(i) Eight groups that are the orthogonal products of the groups 3, $3m$, 6 and $6mm$ by the groups 4 and $4mm$.

(ii) The three groups 64, $64.m$ (with two mirrors) and 64.2.

(iii) Two groups of rotations, the products of the groups 3 and 6 with group 422; six groups as the product of groups 3, $3m$ and 6 with groups $\bar{4}$ and $\bar{4}2m$; and finally the three groups $\bar{3}.4$, $\bar{3}m.4$ and $\bar{3}m.\bar{4}$.

5.2. The (hexagon square)-al family (No. XX) of space E^5 and the WPV symbols of its point groups

The cell of the (hexagon square)-al family is a right hyperprism whose basis is the cell of system 18_1. As previously, we introduce the reflection m_u, which is combined with the symmetry point operations of the system 18_1 groups. Thus, we obtain new symmetry operations such as rotations of order 2 or *homotheties* $\bar{1}$, $\bar{1}_4$ or $\bar{1}_5$.

This family splits into two subfamilies: subfamily XXa and a primitive subfamily XX. The computer-based method gives 119 point groups. Among these, 31 belong to the centred subfamily XXa and 88 to the primitive subfamily XX. The subfamily XXa holohedry is group $\bar{3}m\perp 4mm$ of order 96 = 12 × 8, hence its name, the trigonal square subfamily, and the subfamily XX holohedry is group $6mm\perp 4mm\perp m$ of order 192 = 12 × 8 × 2. Most of the WPV symbols are obtained by studying the elements of the group. The symbols of the 119 point groups must contain one and only one operation of symmetry 3 or 6 and one and only one element of symmetry 4 such as 64 or $3\perp 4$. It is also possible to multiply these elements by operations of order 2. In order to find these symbols, we have on the one hand listed all the point groups of the hexagon square family of space E^4, then combined them with

Table 8
The hexagon square family (system 18_1, space E^4).

All these groups act in the space ($xyzt$). Rotations 3 and 6, groups $3m$ and $6mm$ act in the plane (xy) whereas rotation 4, group $4mm$ acts in the plane (zt). Groups $\bar{3}$, $\bar{3}m$ act in the space (xyz) and groups 422, $\bar{4}$, $\bar{4}2m$ act in the space (ztx). The double rotation 64 belongs to the space ($xyzt$) and rotation 2 acts in the plane (zu).

System 18_1 (22 point groups)			
$E^2 + E^2$	E^4	E^4	E^4
$3\perp 4$	3.422	$3.\bar{4}$	$3.\bar{4}2m$
$3\perp 4mm$			
$3m\perp 4$	$\bar{3}m.4$	$3m.\bar{4}$	$3m.\bar{4}2m$
$3m\perp 4mm$	$\bar{3}.4$	$\bar{3}.\bar{4}$	
$6\perp 4$	6.422	$6.\bar{4}$	$6.\bar{4}2m$
$6\perp 4mm$	64	$64.m$	64.2
$6mm\perp 4$			
$6mm\perp 4mm$ (hol.)			

group m (segment point group), and on the other hand combined all elements of the hexagon and square groups with the operation m to construct new groups.

5.2.1. The trigonal square subfamily (No. XXa) of space E^5. In order to give a WPV symbol to the 31 point groups of subfamily XXa, we give priority to the threefold symmetry. These groups are listed in the first part of Table 9. We find:

(i) Seven point groups belonging to space E^4. Four of them are the direct and orthogonal ($\perp$) products of groups 3 and $3m$ with groups 4 and $4mm$, and three of them are the non-direct (.) products of groups 3 with groups 422, $\bar{4}$ and $\bar{4}2m$ (first column).

(ii) Six groups that are the direct ($\perp$) products of groups (3 2), $\bar{3}$ and $\bar{3}m$ with groups 4 and $4mm$ (second column).

(iii) Two groups $\overline{\overline{34}}$ and $\overline{\overline{34}}.m$. Group $\overline{\overline{34}}$ is a cyclic group of order 12 generated by the point operation $34\times\bar{1}_5$ or $64.m$. According to Hermann–Mauguin conventions, the group $\overline{\overline{34}}.m$ contains two operations m (third and fifth columns).

(iv) Four groups that are the products of groups 3 and $3m$ with groups 42 and $42.m$ (third and fifth columns).

(v) Eight groups that are the products of groups $\bar{3}$, 62 and (3 2) with groups 4, 422, $\bar{4}$, $\bar{4}2m$ and $\bar{\bar{4}}$. Among the 15 groups expected, there are only eight new groups, which are the subgroups of the family holohedry (third, fourth and fifth columns).

(vi) Finally, four groups that are the products of one of the previous groups with the operations $\bar{1}$, $\bar{1}_4$ and $\bar{1}_5$ (third and fourth columns).

5.2.2. The (hexagon square)-al subfamily (No. XX) of space E^5. The majority of the 88 point groups of the subfamily XX are geometrically forseeable. They are listed in the second part of Table 9. In order to asssign a WPV symbol to these groups, we give priority to the sixfold symmetry. All the subgroups of the holohedry $6mm\perp 4mm\perp m$ that have not been listed yet belong to this family. Let us recall that the hexagon square family of space E^4 has 22 point groups and we denote this set by G. Among this set, seven groups belong to the subfamily XXa. We find:

(i) Fifteen point groups among the 22 point groups of the set G (first column).

(ii) Twenty-two point groups $g\perp m$, where g belongs to the set G (second column).

(iii) Fourteen point groups: 12 are direct and orthogonal ($\perp$) products of groups 3, $3m$, 6 and $6mm$ with groups 422, $\bar{4}$ and $\bar{4}2m$, and two are direct products of group 622 with groups 4 and $4mm$ (third column).

(iv) Eleven groups which are the products of groups 6, 622, $\bar{3}$, $\bar{3}m$ and 62 with groups 422, $\bar{4}$, $\bar{4}2m$, 42 and $\bar{\bar{4}}$. Among the 25 groups expected there are only 11 new groups. These are the subgroups of the family holohedry $4mm\perp 4mm\perp m$, but they are not subgroups of the subfamily XXa holohedry $\bar{3}m\perp 4mm$ (fourth, fifth and sixth columns).

(v) Finally, 26 groups that are the products of one of the previous groups with the operations m, 2, $\bar{1}$, $\bar{1}_4$ and $\bar{1}_5$.

Consequently, every WPV point-group symbol of family XX can contain one and only one element of symmetry 3 or 6 and one and only one element of symmetry 4, for example $3\perp 4$, and one element of double symmetry 3 or 6 and 4, such as 64 or $\overline{\overline{34}}$. It is possible for other factors with symmetries of order 2 exclusively to appear, such as m, 2, $\bar{1}$, $\bar{1}_4$ and $\bar{1}_5$.

Conversely, every point group of space E^5 having the previous property belongs to the (hexagon square)-al family, and if it acts in space E^4 then it belongs to the hexagon square family.

6. Conclusion

This paper shows the very important properties of the WPV geometrical names of the crystal families and the WPV symbols of their point groups in any spaces.

The geometrical name of the crystal family gives the symbol of its holohedry, hence its order. Therefore, WPV point-group symbols allow us to know immediately the order and the elements of these point groups. For a few point groups, the note above the table provides help if necessary. For instance, the holohedry of the (di hexagons)-al family is $6mm\perp 6mm\perp m$ of order $12 \times 12 \times 2 = 288$ because the name indicates a right hyperprism, suffix -al, and the subcell is built from two hexagons in two orthogonal spaces.

Moreover, this paper gives a striking example of the importance of crystallography in the plane (E^2) and in physical space (E^3) when studying crystallography in the superspaces E^4, E^5 *etc*. This assertion can be verified as shown for the set of crystal subfamilies in Table 10.

APPENDIX *A*

In this appendix, we show how we can build crystal families of space E^5 from crystal families of spaces E^2, E^3 and E^4 through an example, the (square oblique)-al crystal family (No. X).

Table 11 shows how it is possible to obtain the point groups of the tetragonal, square oblique and square triclinic families from the square family. Table 12 gives a summary of the point groups of these four families.

Table 9
The (hexagon square)-al family (No. XX, space E^5).

The holohedry symbol of subfamily XX is $6mm\perp4mm\perp m$; group $6mm$ acts in the plane (xy), group $4mm$ in the plane (zt) and group m in the space u. The generators for the groups belonging to the hexagon square family are described in Table 8. Here we give some explanations for other groups. Rotation 2: plane (zu), one exception for group $(3\perp4).2$, rotation 2 acts in the plane (xu). *Homothetie* $\bar{1}$: space (xzu). *Homothetie* $\bar{1}_4$: space ($xztu$), one exception for group $(3\perp4).\bar{1}_4$, group $\bar{1}_4$ acts in the space ($xyzu$). Groups (3 2) and 622 act in the space (xyu). Group 62 acts in the space ($xyzu$) even for group 62.42. Group 42 acts in the space ($ztxu$) except for group 62.42.

Subfamily XXa (31 point groups)

E^4	$E^3 + E^2$	E^5	E^5	E^5
$3\perp4$	$(3\ 2)\perp4$		$(3\perp4).\bar{1}$	
$3\perp4mm$	$(3\ 2)\perp4mm$	$(3\ 2).\bar{4}$		$(3\ 2).\bar{\bar{4}}$
$3m\perp4$		$(3m\perp4).\bar{1}$		
$3m\perp4mm$		$3m.42$		$3m.42.m$
		$\overline{\overline{34}}$		$\overline{\overline{34}}.m$
3.422		3.42	$3.42.\bar{1}_4$	$3.42.m$
	$\bar{3}\perp4$	$\bar{3}.422$		
	$\bar{3}\perp4mm$			
	$\bar{3}m\perp4$	62.4	62.422	$62.\bar{4}$
	$\bar{3}m\perp4mm$ (hol.)			
$3.\bar{4}$		$(3.\bar{4})\times\bar{1}_5$		
$3.\bar{4}2m$		$\bar{3}.\bar{4}2m$		$62.\bar{4}2m$

Subfamily XX (88 point groups)

E^4	$E^4 + E^1$	$E^3 + E^2$	E^5	E^5	E^5
	$3\perp4\perp m$		$(3\ 2).42$		
	$3\perp4mm\perp m$				
	$(3.422)\perp m$	$3\perp422$	$(3\perp422).\bar{1}_4$		$(3\perp422)\times\bar{1}_5$
		$3\perp\bar{4}$	$(3\perp\bar{4}).2$	$(3\perp\bar{4}).\bar{1}_4$	$(3\perp\bar{4}).\bar{1}$
					$(3\perp\bar{4})\times\bar{1}_5$
	$(3.\bar{4})\perp m$		$3.\bar{4}.\bar{1}$		
	$(3.\bar{4}2m)\perp m$	$3\perp\bar{4}2m$	$(3\perp\bar{4}2m).2$		$(3\perp\bar{4}2m)\times\bar{1}_5$
	$3m\perp4\perp m$	$3m\perp422$			
	$3m\perp4mm\perp m$	$3m\perp\bar{4}$			$(3m\perp\bar{4})\times\bar{1}_5$
		$3m\perp\bar{4}2m$			$(3m\perp\bar{4}2m)\times\bar{1}_5$
$6\perp4$	$6\perp4\perp m$	$6\perp422$	$(6\perp4).\bar{1}$		
$6\perp4mm$	$6\perp4mm\perp m$				
$6mm\perp4$	$6mm\perp4\perp m$	$6mm\perp422$			
$6mm\perp4mm$	$6mm\perp4mm\perp m$ (hol.)				
		$6\perp\bar{4}$	$(6\perp\bar{4}).\bar{1}$		
		$6\perp\bar{4}2m$	6.42		$6.42.m$
		$6mm\perp\bar{4}$	$622.\bar{4}$		
		$6mm\perp\bar{4}2m$			
		$622\perp4$	622.422		622.42
		$622\perp4mm$	$622.\bar{4}2m$		62.42
64	$64\perp m$		$64.\bar{1}$		$64.\bar{1}_4$
64.2	$(64.2)\perp m$		$64.2.\bar{1}_4$		$(64.2)\times\bar{1}_5$
$64.m$	$(64.m)\perp m$		$64.m.\bar{1}$		
$3m.\bar{4}$	$(3m.\bar{4})\perp m$		$3m.\bar{4}.\bar{1}$		
$3m.\bar{4}2m$	$(3m.\bar{4}2m)\perp m$				
$\bar{3}.4$	$(\bar{3}.4)\perp m$		$\bar{3}.4.\bar{1}$	$\bar{3}.42$	$(\bar{3}.42)\times\bar{1}_5$
6.422	$(6.422)\perp m$				
$6.\bar{4}$	$(6.\bar{4})\perp m$		$6.\bar{4}.\bar{1}$		
$6.\bar{4}2m$	$(6.\bar{4}2m)\perp m$				
$\bar{3}m.4$	$(\bar{3}m.4)\perp m$		$\bar{3}m.422$		
$\bar{3}.\bar{4}$	$(\bar{3}.\bar{4})\perp m$		$\bar{3}.\bar{4}.\bar{1}_4$		$\bar{3}m.\bar{4}$
			$\bar{3}.\bar{\bar{4}}$	$\bar{3}.\bar{\bar{4}}.2$	$\bar{3}m.\bar{\bar{4}}$

Table 10
Relations between subfamilies in different spaces.

Space	Family name	Holohedry	Subfamily	Holohedry
E^2	hexagon	$6mm$		
E^3	hexagonal	$6mm\perp m$	hexagonal	$6mm\perp m$
			trigonal†	$\bar{3}m$
E^4	di hexagons	$6mm\perp 6mm$	20_3	$6mm\perp 6mm$
			20_2	$3m.\bar{3}m$
			20_1	63.222
E^5	(di hexagons)-al	$6mm\perp 6mm\perp m$	XXI	$6mm\perp 6mm\perp m$
			XXIa	$6mm\perp\bar{3}m$
			XXIb	$(3m.\bar{3}m)\perp m$
			XXIc	$(3m\perp 3m)\times\bar{1}_5$
			XXId	$\bar{3}.\bar{3}m$
			XXIe	$(3\perp 3m)\times\bar{1}_5$
			XXIf	$\{(3\,2).3\}\times\bar{1}_5$
			XXIg	$(3\perp 3)\times\bar{1}_5$
			XXIh	$(63.2)\times\bar{1}_5$
E^6	di hexagons square	$6mm\perp 6mm\perp 4mm$		
E^6	tri hexagons	$6mm\perp 6mm\perp 6mm$		
E^7	(tri hexagons)-al	$6mm\perp 6mm\perp 6mm\perp m$		
E^8	hexagon square oblique rectangle	$6mm\perp 4mm\perp 2\perp mm$		

† The whole Hermann–Mauguin holohedry symbol is $\bar{3}2/m$.

Table 11
From the square family to the tetragonal, square oblique and square triclinic families.

Point symmetry operation is abbreviated as PSO.

Tetragonal family (7 point groups)			Square oblique family (7 point groups)			Square triclinic family (7 point groups)		
PSOs	4	4; m_x	PSOs	4	4; m_x	PSOs	4	4; m_x
1	4	$4mm$	1	4	$4mm$	1	4	$4mm$
m_z	$4\perp m$	422	2_{zt}	42	$42.m$		$4\perp\bar{1}$	$4mm\perp\bar{1}$
	$\bar{4}$	$\bar{4}2m$	$4\perp 2$	$4mm\perp 2$			$\bar{\bar{4}}$	$\bar{\bar{4}}.m$
		$4mm\perp m$		$4.\bar{1}$				$4.\bar{1}_4$

Table 12
Point groups of the square, tetragonal, square oblique and square triclinic families.

Square (E^2)	4	$4mm$					
Tetragonal (E^3)	4	$4mm$	$\bar{4}$	$\bar{4}2m$ ($\bar{4}.m$)	422 (4.2)	$4\perp m$	$4\perp mmm$
Square oblique (E^4)	4	$4mm$	42	$42.m$ ($42\bar{1}m$)	$4.\bar{1}$ ($4\bar{1}\,\bar{1}$)	$4\perp 2$	$4mm\perp 2$
Square triclinic (E^5)	4	$4mm$	$\bar{\bar{4}}$	$\bar{\bar{4}}.m$ ($\bar{\bar{4}}\,\bar{1}_4m$)	$4.\bar{1}_4$ ($4\bar{1}_4\bar{1}_4$)	$4\perp\bar{1}$	$4mm\perp\bar{1}$

Table 13
The (square oblique)-al family (No. X, space E^5).

Group 2 acts in the plane (zt). Groups 4 and $4mm$ act in the plane (xy) and reflection m acts in the axis u. Groups $\bar{4}$, 422 and $\bar{4}2m$ act in the space (xyu). *Homothetie* $\bar{1}$ acts in space (xzt). *Homothetie* $\bar{1}_4$ acts in space ($xztu$).

Subfamily Xa (8 point groups)			Subfamily X (16 point groups)				
E^3	E^4	E^5	E^3	$E^3 + E^2$	E^4	$E^4 + E^1$	E^5
$\bar{4}$		$\bar{4}.\bar{1}_4$		$\bar{4}\perp 2$			$\bar{\bar{4}}.\bar{1}$
$\bar{4}2m$		$\bar{4}2m\times\bar{1}_5$ (hol.)		$\bar{4}2m\perp 2$			
	42	$42\times\bar{1}_5$				$42\perp m$	
	$42.m$	$42.\bar{1}_4$				$(42.m)\perp m$	
			422	$422\perp 2$			$422\times\bar{1}_5$
			$4\perp m$		$4\perp 2$	$4\perp 2\perp m$	
			$4mm\perp m$		$4mm\perp 2$	$4mm\perp 2\perp m$ (hol.)	
					$4.\bar{1}$	$(4.\bar{1})\perp m$	

The cell of the (square oblique)-al crystal family can be built:

(i) from the cell of the square oblique crystal family of space E^4. We add an axis u orthogonal to space E^4, hence we obtain the reflection m_u.

(ii) or from the cell of the tetragonal crystal family of space E^3, square in the plane (xy) and with axis u. We add the plane (zt) orthogonal to plane (xy) and to axis u. Hence we add a rotation of order 2 in the plane (zt) and with reflection m_u we generate the *homothetie* $\overline{1}_{ztu}$.

The 24 point groups of the (square oblique)-al crystal family are listed Table 13. This family is divided into two subfamilies:

(i) Subfamily Xa. This has eight point groups and group $\overline{4}2m\times\overline{1}_5$ of order 16 for holohedry.

(ii) Subfamily X. This has 16 point groups and group $4mm\perp2\perp m$ of order 32 for holohedry.

In Table 13, we find:

(i) Five point groups of the tetragonal family, denoted g, *i.e.* $\overline{4}$, $\overline{4}2m$, 422, $4\perp m$ and $4mm\perp m$ (first and fourth columns). Groups 4 and $4mm$ belong to the square triclinic family of space E^5.

(ii) Five point groups of the square oblique crystal family, denoted g', *i.e.* 42, $42.m$, $4.\overline{1}$, $4\perp2$ and $4mm\perp2$ (second and sixth columns) together with the five groups $g'\perp m$ (seventh column).

(iii) Three groups $g\perp2$ that do not belong to the previous columns, *i.e.* $\overline{4}\perp2$, $\overline{4}2m\perp2$ and $422\perp2$ (fifth column).

(iv) Six groups belonging to space E^5 (third and eighth columns). Three of them are of type $g\times\overline{1}_5$ or $g'\times\overline{1}_5$, *i.e.* groups $422\times\overline{1}_5$, $\overline{4}2m\times\overline{1}_5$ and $42\times\overline{1}_5$.

The other three groups are obtained as follows:

(i) The elements of group $\overline{\overline{4}}$ (square triclinic family) are the following: 1, 2_{xy} and $4_{xy}^{\pm1}\overline{1}_{ztu}$. The product of the reflection m_{x-y} (an element of the square point group) with the rotation 2_{zt} results in the homothetie $\overline{1}_{x+y\,zt}$. The products of this *homothetie* with the elements of the group $\overline{\overline{4}}$ give the following operations: 2_{xu}, 2_{yu} and $\overline{1}_{x-y\,zt}$. Hence we obtain a new group denoted $\overline{\overline{4}}.\overline{1}$.

(ii) Group 42 of the square oblique crystal family can be combined with an operation of the point group of the square triclinic family, $\overline{1}_{x+y\,ztu}$, for example. The group obtained has the symbol $42.\overline{1}_4$.

(iii) We continue this process from group $\overline{4}$ and *homothetie* $\overline{1}_{x+y\,ztu}$ and we obtain group $\overline{4}.\overline{1}_4$.

References

Brown, H., Bülow, R., Neubüser, J., Wondratschek, H. & Zassenhaus, H. (1978). *Crystallographic Groups of Four-Dimensional Space*. New York: John Wiley.

International Tables for Crystallography (1995). Vol. A, *Space-Group Symmetry*, 4th revised ed., edited by Th. Hahn. Dordrecht: Kluwer Academic Publishers.

Janssen, T., Birman, J. L., Dénoyer, F., Koptsik, V. A., Verger-Gaugry, J. L., Weigel, D., Yamamoto, A., Abrahams, S. C. & Kopsky, V. (2002). *Acta Cryst.* A**58**, 605–621.

Janssen, T., Birman, J. L., Koptsik, V. A., Senechal, M., Weigel, D., Yamamoto, A., Abrahams, S. C. & Hahn, T. (1999). *Acta Cryst.* A**55**, 761–782.

Plesken, W. (1981). *Match*, **10**, 97–119.

Plesken, W. & Hanrath, W. (1984). *Math. Comput.* **43**, 573–587.

Veysseyre, R. & Veysseyre, H. (2002). *Acta Cryst.* A**58**, 429–433.

Veysseyre, R., Weigel, D., Phan, Th. & Veysseyre, H. (1998). *Advances in Structure Analysis*, edited by R. Kužel & J. Hašek, pp. 1–15. Prague: Czech and Slovak Crystallographic Association.

Veysseyre, R., Weigel, D., Phan, T. & Veysseyre, H. (2002). *Acta Cryst.* A**58**, 434–440.

Weigel, D., Phan, T. & Veysseyre, R. (1987). *Acta Cryst.* A**43**, 294–304.

Weigel, D., Phan, Th. & Veysseyre, R. (2008). *Acta Cryst.* A**64**, 687–697.

Weigel, D. & Veysseyre, R. (1993). *Acta Cryst.* A**49**, 486–492.